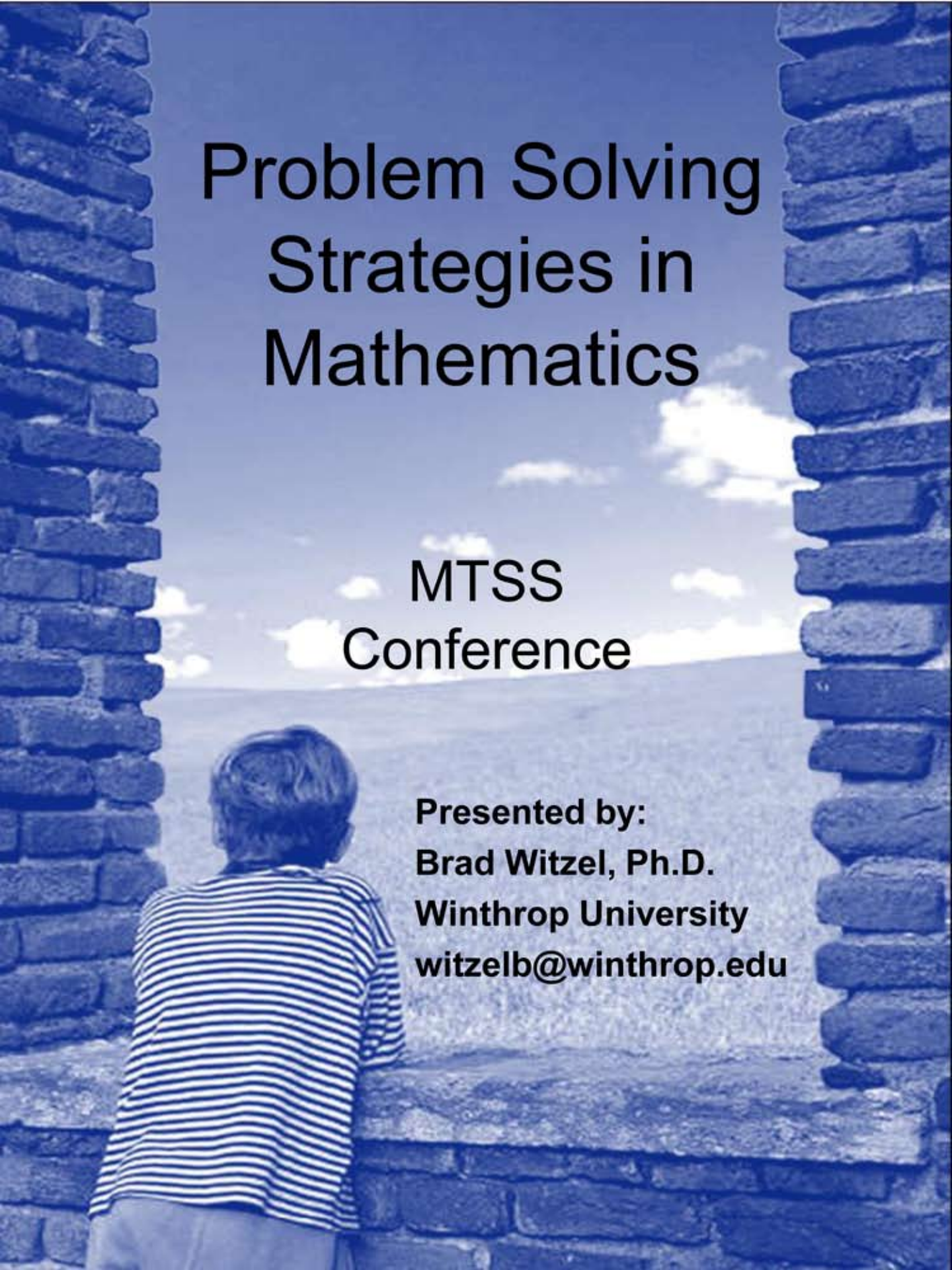

Problem Solving Strategies in Mathematics

MTSS Conference

Presented by:
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Mathematics Problem Solving: Schema-based and Structured Strategies

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90min

 1

Agenda

- Review SBAC-structured questions
- Establish a preparatory approach
  1) **Increase frequency and type of problem solving activities**
  2) **Teach students meta-thinking during problem solving**
  3) **Model visuals and charting of word problems**
  4) **Teach schema-based and problem-solving approaches**
  5) **Use math language during teacher and student think alouds**

 2

4th grade item – area + perimeter

Ms. McCrary wants to make a rabbit pen in a section of her lawn. Her plan for the rabbit pen includes the following:

- It will be in the shape of a rectangle.
- It will take 24 feet of fence material to make.
- Each side will be longer than 1 foot.
- The length and width will measure whole feet.

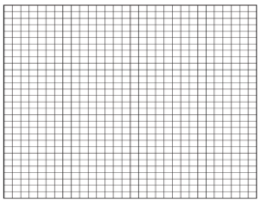
 3

4th grade item – area + perimeter

Part A

Draw 3 **different** rectangles that can each represent Ms. McCrary's rabbit pen. Be sure to use all 24 feet of fence material for each pen.

Use the grid below. Click the places where you want the corners of your rectangle to be. Draw one rectangle at a time. If you make a mistake, click on your rectangle to delete it. Continue as many times as necessary.



Key
☐ = 1 square foot

 4

4th grade item – area + perimeter

Use your keyboard to type the length and width of each rabbit pen you draw. Then type the area of each rabbit pen. Be sure to select the correct unit for each answer.

[Students will input length, width, and area for each rabbit pen. Students will choose unit from drop down menu.]

Pen 1

Length: ☐ ☐ (feet, square feet)
Width: ☐ ☐ (feet, square feet)
Area: ☐ ☐ (feet, square feet)

Pen 2

Length: ☐ ☐ (feet, square feet)
Width: ☐ ☐ (feet, square feet)
Area: ☐ ☐ (feet, square feet)

Pen 3

Length: ☐ ☐ (feet, square feet)
Width: ☐ ☐ (feet, square feet)
Area: ☐ ☐ (feet, square feet)

 5

4th grade – deciphering a table

A scientist watched a group of squirrels collect acorns. Each squirrel **ate** some of the collected acorns and **stored** the rest of the collected acorns.

The table below shows data for three squirrels in the group. The number of acorns each squirrel **stored** is missing from the table. Fill in the data that are missing from the table.

Acorns Collected by Squirrels

Squirrel	Number Eaten	Number Stored	Total Number Collected
X	40		100
Y	50		105
Z	35		95

 6

5th grade - measurement

Xian is practicing his long jumps for the track team.

- His first jump measured 3 yards, 1 foot, 2 inches.
- His second jump measured 2 yards, 2 feet, 9 inches.

How much farther is Xian's first jump than his second jump? Be sure to include the measurement units.

 7

5th grade - computation

Mrs. Phelps bought 4 boxes of crayons at the store to share with her students. Each box contained a total of 64 crayons.

Part A

What is the total number of crayons Mrs. Phelps bought at the store? Explain your answer using diagrams, pictures, mathematical expressions and/or words.

☐ crayons

Part B

Mrs. Phelps wants to give each of her students an equal number of the crayons she bought. There are 32 students in Mrs. Phelps' class. How many crayons should each student get?

☐ crayons

Part C

How many **more** boxes of crayons does Mrs. Phelps need if she wants each of her students to get 12 crayons? Explain your answer using diagrams, pictures, mathematical expressions and/or words.

☐ boxes of crayons

 8

6th grade – ratio and proportions

Ben's Game World is having a sale on video games. The store is offering a sale pack of 4 video games for $43.80. What is the unit price of a video game in the sale pack?

$ ☐

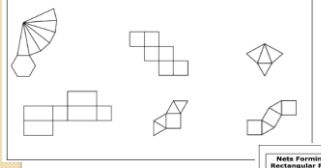
Roberto's Electronics is also having a sale on video games. The unit price of any video game at Roberto's Electronics is the same as the unit price of a video game in the sale pack at Ben's Game World. How much would it cost a customer for 7 video games at Roberto's Electronics?

$ ☐

 9

6th grade – geometrical nets

Classify each net as representing a rectangular prism, a triangular prism, or a pyramid. To place an object in a region, click the object, move the pointer over the region, and click again to place the object in the region. To return all objects to their original positions, click the Reset button.



Nets Forming a Rectangular Prism	Nets Forming a Triangular Prism	Nets Forming a Pyramid

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Algebra – Q2, part 1

Your Assignment:
In this task you will assume the role of consultant for a farmer. You will analyze the options available to the farmer for handling the storage of shelled field corn (shown in the pictures above). In the past, the farmer has sold the corn as it was harvested, and did not store the corn to be sold in the future. The farmer has increased the number of acres used to grow corn, and now is exploring the cost of storing the corn until after the harvest is complete and then selling it. You will analyze two storage options available to the farmer for storing the grain that is harvested.

- The corn can be stored in grain bins constructed on the farm.
- The corn can be stored in rental storage close to the farm.

Following the tasks, you will recommend which type of storage the farmer should use.

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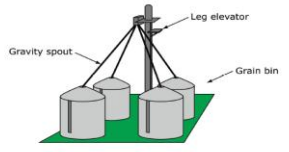
Algebra – Q2, part 2

Corn Storage

Session 1

Part A

Your first job is to determine the most efficient cost for constructing 4 grain bins which can be used to store the harvested corn. A leg elevator, which moves the corn from ground level into the bins, must also be built. The bins must be able to hold the 132,000 bushels of corn from the harvest. Each bin should include a ventilated floor, fan and heat. A cost table for building various size bins is shown below.



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Algebra-Q2, part3

Cost of Grain Bins

Diameter (feet)	Height (feet)	Capacity (bushels)	Cost Without Floor ($)	Add for			
				Concrete Floor ($)	Steel Floor ($)	Ventilated Floor ($)	Fan and Heat ($)
30	19	10,775	11,525	1,650	1,130	4,250	2,950
	24	13,625	16,080	1,775	1,130	4,250	2,950
	32	18,175	20,180	1,975	1,130	4,250	2,950
33	24	16,475	17,725	2,050	1,320	5,100	3,025
	27	18,550	20,050	2,100	1,320	5,100	3,025
	32	21,975	24,950	2,550	1,320	5,100	3,025
36	24	19,625	21,575	2,575	1,540	6,000	3,075
	27	22,075	23,475	2,675	1,540	6,000	3,075
	32	26,150	26,100	2,775	1,540	6,000	3,075
	40	32,700	28,925	2,950	1,540	6,000	3,075
42	27	30,050	28,450	3,650	2,065	8,100	3,225
	32	35,600	32,525	3,875	2,065	8,100	3,225
	40	44,500	39,650	4,075	2,065	8,100	3,225
	48	53,425	47,280	4,400	2,065	8,100	3,225
48	27	39,250	41,150	4,775	2,640	10,450	3,350
	32	46,500	48,980	5,050	2,640	10,450	3,350
	40	58,150	55,000	5,300	2,640	10,450	3,350
	48	69,775	61,650	5,750	2,640	10,450	3,350

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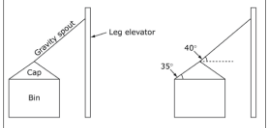
Algebra, Q2, part4

All 4 bins must have the same capacity.

The bins must be built to the following specifications.

- The height listed in the table does not include the height of the conical cap on top of the bin. The cap forms a 35° angle with the base.
- The distance from the outer edge of the bins to the leg elevator will be 15 feet.
- A gravity spout is placed so that it runs from the top of the cap to a point that is 2 feet below the top of the elevator leg. To account for certain moisture content the gravity spouts will slope 40 degrees to the horizontal.
- The average cost involved in the construction of the leg elevator is $15,000 plus $125 for every foot in height.
- The gravity spouts cost $20 per foot.



Find the most efficient cost of the construction. Be sure to include the bins (caps are included in the price), gravity spouts, and leg elevator.

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Algebra, Q2, part 5

Part B

When corn is harvested, the moisture content varies, but is typically above the level desired for selling or storing corn, so the corn must be dried. The moisture content is given as a percent that represents the proportion of the weight of the corn that is from water. For example, if you had 40 lbs of corn at 25% moisture content, it would consist of 10 lbs of water and 30 lbs of dry material. As corn dries, the amount of water decreases, but the amount of dry material stays the same, so the percent of weight from water will decrease.

The final moisture contents for various purposes are as follows:

- 15.5% to sell at market
- 14.0% to store at a rental storage facility
- 13.5% to store in grain bins constructed on the farm

There is a cost for drying corn to 15.5% moisture to be able to sell it at market, but there is extra cost to dry it below 15.5%. This extra cost is a cost of storage since it must be paid only if the grain is to be stored and not sold at market.

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Algebra, Q2, part 6

Assuming corn is harvested at an initial moisture content of 20% and you use LP gas as fuel for your dryer, use the information in tables 1 and 2 below to calculate the extra cost per bushel of drying corn to a final moisture content of 14% and 13.5%. Justify your answer mathematically and show all the steps in your calculation. You can use the regression tool in the spreadsheet provided if necessary. The BTUs required to dry corn to a final moisture content of 15.5% and 13.5% are not in the table but can be found using the provided regression tool.

Energy (BTU's) Required to Dry a Bushel of Wet Corn

Final Moisture Content	Initial Moisture Content					
	20%	22%	24%	26%	28%	30%
17%	5,625	8,744	11,714	14,487	17,086	19,545
16%	7,522	10,596	13,506	16,241	18,784	21,188
15%	9,579	12,589	15,447	18,118	20,624	22,978
14%	11,635	14,582	17,388	19,994	22,463	24,768
13%	13,878	16,774	19,528	22,088	24,486	26,744

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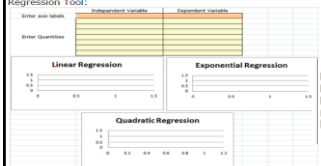
Algebra, Q2, part 7

Energy Content (BTU's) per Unit of Fuel

Fuel Type	Unit	BTU's per Unit of Fuel
Oil	Gallon	140,000
LP gas	Gallon	92,000
Electricity	kWh	3,414
Natural gas	Cubic foot	1,000

To use the regression tool below, enter labels for the axes and pairs of independent and dependent variable values in the spreadsheet.

Regression Tool:



Enter your final answers:
Extra cost to dry 1 bushel of corn to 14% = _____
Extra cost to dry 1 bushel of corn to 13.5% = _____
(Record these values on your note sheet; you will need them in a later part.)

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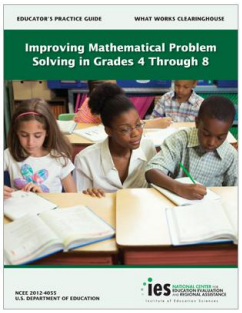
IES Practice Guide on Math Problem Solving



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Improving Mathematical Problem Solving Practice Guide (Woodward et al., 2012)

- **Recommendation 1**. Prepare problems and use them in whole-class instruction
- **Recommendation 2**. Assist students in monitoring and reflecting on the problem-solving process
- **Recommendation 3**. Teach students how to use visual representations
- **Recommendation 4**. Expose students to multiple problem-solving strategies
- **Recommendation 5**. Help students recognize and articulate mathematical concepts and notation

Recommendation 1

- *"Problem solving must be an integral part of each curricular unit, with time allocated for problem-solving activities with the whole class. In this recommendation, the panel provides guidance for thoughtful preparation of problem-solving lessons. Teachers are encouraged to use a variety of problems intentionally and to ensure that students have the language and mathematical content knowledge necessary to solve the problems" (Woodward, et al., 2012, p. 10).*

Daily: Use problems that are routine and non-routine

1. Carlos has a cake recipe that calls for 2 3/4 cups of flour. He wants to make the recipe 3 times. How much flour does he need? *This problem is likely routine for a student who has studied and practiced multiplication with mixed numbers.*
2. Solve $2y + 15 = 29$ *This problem is likely routine for a student who has studied and practiced solving linear equations with one variable.*
3. Two vertices of a right triangle are located at (0,4) and (0,10). The area of the triangle is 12 square units. Find a point that works as the third vertex. *This problem is likely routine for a student who has studied and practiced determining the area of triangles and graphing in coordinate planes.*

Present non-routine problems

a) There are 20 people in a room. Everybody high-fives with everybody else. How many high-fives occurred?
This problem is likely non-routine for students in beginning algebra.

b) Solve for the variables *a* through *f* in the equations below, using the digits from 0 through 5. Every digit should be used only once. A variable has the same value everywhere it occurs, and no other variable will have that value.
$a + a + a = a^2$; $b + c = b$; $d \times e = d$; $a - e = b$; $b^2 = d$; $d + e = f$
The problem is likely non-routine for a student who has not solved equations by reasoning about which values can make an equation true.

c) In a leap year, what day and time are exactly in the middle of the year?
This problem is likely non-routine for a student who has not studied problems in which quantities are subdivided into unequal groups.

Clarify Vocabulary and Context of Problems

Example Problem	Vocabulary	Context
In a factory, 54,650 parts were made. When they were tested, 4% were found to be defective. How many parts were working?	Students need to understand the term *defective* as being the opposite of *working* and the symbol % as *percent* to correctly solve the problem.	What is a *factory*? What does *parts* mean in this context?
At a used-car dealership, a car was priced at $7,000. The salesperson then offered a discount of $350. What percent discount, applied to the original price, gives the offered price?	Students need to know what *offered* and *original price* mean to understand the goal of the problem, and they need to know what *discount* and *percent discount* mean to understand what mathematical operators to use.	What is a *used-car dealership*?

Beware textbook definitions: Fraction's example

- A number used to name a part of a group or a whole (math.com)
- A math expression with two numbers placed above and below a division line indicating the number of divisions or portions and the size of each division or portion (toolingu.com)
- A fraction is a part or portion of a whole (kidsorl.com)
- An expression that indicates the quotient of two quantities (freedictionary.com)

Reading Levels of Math Texts in ES

Grade	Textbook publisher	Area of Textbook	Directions readability range	Word Problems readability range
3	Popular textbook program 1	2-digit addition, perpendicular lines, lines, angles, and fractions	2nd -5th	4th – 6th
3	Popular textbook program 2		6th – 8th	2nd – 6th
5	Popular textbook program 1	solution and least common multiple	8th – 9th	4th – 6th
5	Popular textbook program 2	solution and least common multiple	6th – 7th	3rd – 6th

Beware textbook definitions: MS

- Diameter is a line segment that passes through the center of a circle and has endpoints on the circle, or the length of that segment.
- Compensation: When a number in a problem is close to another number that is easier to calculate with, the easier number is used to find the answer. Then the answer is adjusted by adding to it or subtracting from it.

Sample definitions from major math textbooks: Middle School

- Polyhedron: A polyhedron is a three-dimensional object, or solid figure, with flat surfaces, called faces, that are polygons.
- Upper quartile: the median of the upper half of the data; also called third quartile.

Math reading confusion in MS

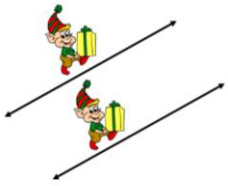
6th grade readability (variables, expression, least common multiple and stem and leaf plot.)	Directions grade level range	Word Problem grade level range
Popular Textbook program 1	4th-10th	5th-7th
Popular Textbook program 2	5th-11th	5th-9th
Popular Textbook program 3	5th-17th	5th-6th

8th grade readability (coordinate plane and function)	Directions grade level range	Word Problem grade level range
Popular Textbook program 1	7th-10th	6th-7th
Popular Textbook program 2	9th-10th	10th-12th
Popular Textbook program 3	7th-9th	7th-12th

Parallel Lines (Pair of Elves)

Parallel Lines (Pair of Elves)

Lines that are the same distance apart and will never intersect

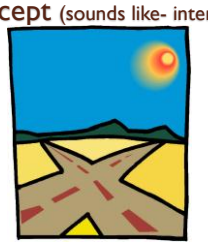


The Pair of Elves are the same distance apart and will never intersect.

The Pair of Elves are on Parallel Lines

Example:

Intercept (sounds like- intersection)



The place where a line, curve or surface crosses an axis

(adapted from Riccomini, 2006)

Recommendation 2

- **Assist students in monitoring and reflecting on the problem-solving process.**
- *"…the panel suggests that teachers help students learn to monitor and reflect on their thought process when they solve math problems. While the ultimate goal is for students to monitor and reflect on their own while solving a problem, teachers may need to support students when a new activity or concept is introduced" (p. 17).*

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Model problem solving prompts

- What is the story in this problem about?
- What is the problem asking?
- What do I know about the problem so far ? What information is given to me? How can this help me?
- Which information in the problem is relevant?
- In what way is this problem similar to problems I have previously solved?
- What are the various ways I might approach the problem?
- Is my approach working? If I am stuck, is there another way I can think about solving this problem?
- Does the solution make sense? How can I verify the solution?
- Why did these steps work or not work?
- What would I do differently next time?

Note: These are examples of the kinds of questions that a teacher can use as prompts to help students monitor and reflect during the problem-solving process. Select those that are applicable for your students, or formulate new questions to help guide your students.

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Math Discourse

Dr. Deborah Ball on math discourse

http://www-personal.umich.edu/~dball/multimedia/index.html

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Recommendation 3

- Teach students how to use visual representations
- Include such visuals as:
  - **Strip diagrams** use rectangles to represent quantities presented in the problem.
  - **Percent bars** are strip diagrams in which each rectangle represents a part of 100 in the problem.
  - **Schematic diagrams** demonstrate the relative sizes and relationships between quantities in the problem.

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Note to Interventionists

"Intervention materials should include opportunities for students to work with visual representations of mathematical ideas and interventionists should be proficient in the use of visual representations of mathematical ideas"

(IES Practice Guide for Assisting Student Struggling with Mathematics, Gersten et al, 2009)

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Ratio problem solving (adapted from Beckman, 2011)

What happens when we mix 3 cups of blue paint with 2 cups of yellow paint?

That was 1 batch. What if we make more batches?

# of batches	1	2	3	4	5	6	7
# cups of blue	3	6	9	12	15	18	21
# cups of yellow	2	4	6	8	10	12	14
# cups of green produced	5	10	15	20	25	30	35

What do these paint mixtures have in common? Name the ratio.

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Ratio problem solving (Adapted from math/about.com)

Manuel is making snowballs during a snowstorm. He can make 40 snowballs in a hour but 5 snowballs melt every 20 minutes. How long will is take to build 220 snowballs

minutes	60	80	100	120	140	160	180
Snowballs	40	80	120	160	200	240	280
Lost to melting	5	10	15	20	25	30	35
Total number of snowballs	35	70	105	140	175	210	245

What trends appear in this problem?

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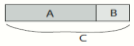
Strip diagrams (Gersten et al., 2009)

The two problems below are addition and subtraction problems that students may be tempted to solve using an incorrect operation. In each case, students can draw a simple diagram like the one shown below, record the known quantities (two out of three of A, B, and C), and then use the diagram to decide whether addition or subtraction is the correct operation to use to determine the unknown quantity.



1) Brad has a bottlecap collection. After Madhavi gave Brad 28 more bottlecaps, Brad had 111 bottlecaps. How many bottlecaps did Brad have before Madhavi gave him more?

2) Brad has a bottlecap collection. After Brad gave 28 of his bottlecaps to Madhavi, he had 83 bottlecaps left. How many bottlecaps did Brad have before he gave Madhavi some?

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Recommendation 4

- **Expose students to multiple problem-solving strategies.**
- Approach (Polya, 1945)
  - Understand the problem
  - Devise a plan
    - **Structure**
    - **Schema-based**
  - Carry out the plan
  - Reflect

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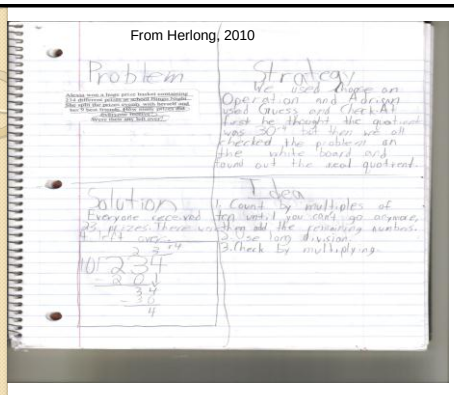
From Herlong, 2010



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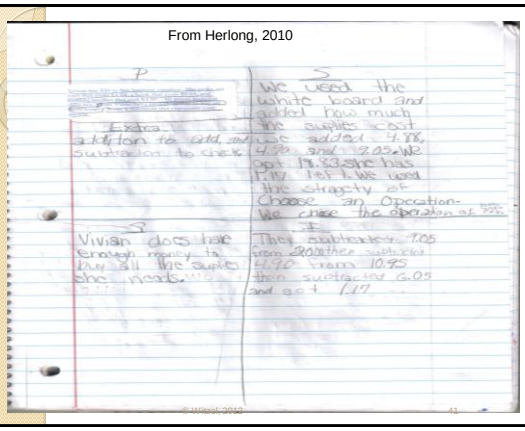
From Herlong, 2010



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Note to Interventionists: Schema-based instructional (SBI) strategies

"Interventions should include instruction on solving word problems that is based on common underlying structure"

(IES Practice Guide for Assisting Student Struggling with Mathematics, Gersten et al, 2009)

Level of Evidence: Strong

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Routine Problem Types: Group; Change; Compare (Jitendra, 2008; Jungjohann, 2010)

Group – two or more smaller groups are grouped to make up a larger group

Change – two or more sequential actions lead to an increase or decrease in total quantity or value

Compare – two items are compared using a common unit or referent

Name and explain the problem type: Group; Change or Compare

a) In a relay race, Caroline ran 6 laps. Her partner ran another 6 laps. How many laps did they run altogether?

b) Marcus read 18 books over the summer. JoJo read 6. How many more did Marcus read?

c) The computer lab has 25 computers. Six more new computers were brought in. How many are there now?

d) The midday temperature was 93 degrees. It dropped 27 degrees overnight. What is the new temperature?

Change Problem (Gersten et al, p.27)

The two problems here are addition and subtraction problems that students may be tempted to solve using an incorrect operation. In each case, students can draw a simple diagram like the one shown below, record the known quantities (two of three of A, B, and C) and then use the diagram to decide whether addition or subtraction is the correct operation to use to determine the unknown quantity.

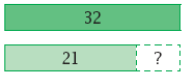
Problem 1. Brad has a bottlecap collection. After Madhavi gave Brad 28 more bottlecaps, Brad had 111 bottlecaps. How many bottlecaps did Brad have before Madhavi gave him more?

Problem 2. Brad has a bottlecap collection. After Brad gave 28 of his bottlecaps to Madhavi, he had 83 bottlecaps left. How many bottlecaps did Brad have before he gave Madhavi some?

Compare problem (Gersten et al, p. 28)

Example 2. Compare problems

There are 21 hamsters and 32 kittens at the pet store. How many more kittens are at the pet store than hamsters?



Recommendation 5

- **Help students recognize and articulate mathematical concepts and notation.**
- Increase verbalization in class, particularly on troublesome content.

Conclusion

- Describe the aspects to SBAC word problems.
- What are the IES problem solving recommendations?
- Describe three main problem types for schema-based problem solving.

References

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